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Prepotential and instanton corrections in $\mathcal{N}=2$ supersymmetric $\mathrm{SU}(N_1)\times\mathrm{SU}(N_2)$ Yang Mills theories

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Abstract: In this paper we analyse the non-hyperelliptic Seiberg-Witten curves derived from M-theory that encode the low energy solution of $\mathcal{N}=2$ supersymmetric theories with product gauge groups. We consider the case of a $\mathrm{SU}(N_1)\times\mathrm{SU}(N_2)$ gauge theory with a hypermultiplet in the bifundamental representation together with matter in the fundamental representations of $\mathrm{SU}(N_1)$ and $\mathrm{SU}(N_2)$. By means of the Riemann bilinear relations that hold on the Riemann surface defined by the Seiberg-Witten curve, we compute the logarithmic derivative of the prepotential with respect to the quantum scales of both gauge groups. As an application we develop a method to compute recursively the instanton corrections to the prepotential in a straightforward way. We present explicit formulas for up to third order on both quantum scales. Furthermore, we extend those results to $\mathrm{SU}(N)$ gauge theories with a matter hypermultiplet in the symmetric and antisymmetric representation. We also present some non-trivial checks of our results.

Keywords: M-Theory, Supersymmetric Effective Theories.

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1. Introduction

It is by now a well-known fact that, as long as $\mathcal{N}=2$ supersymmetry is unbroken, the low energy effective action is given in terms of a holomorphic prepotential $\mathcal{F}$. The Seiberg-Witten solution for this low-energy effective action [1] allows us, in principle, to reconstruct the prepotential of the theory using a set of algebro-geometric data. Such solution is given in terms of a suitable Riemann surface or algebraic curve Σ, and a preferred meromorphic 1-form dS_{SW}, which is known as Seiberg-Witten differential. This differential induces a special geometry on Σ, and its periods give the spectrum of BPS states of the theory. Interestingly enough, this solution displays remarkable nonperturbative phenomena such as quark confinement by monopole condensation, when a mass term that breaks supersymmetry down to $\mathcal{N}=1$ is included.

The original work of Seiberg and Witten [1] was developed for $\mathcal{N}=2$ theories with gauge group SU(2) with and without matter in the fundamental representation. Nevertheless, this solution was soon extended to other gauge groups and matter content by determining both the appropriate complex curve and meromorphic differential [2]–[6], thus leading to a substantial progress in our understanding of $\mathcal{N}=2$ supersymmetric gauge theories. In fact, the appearance of an auxiliary Riemann surface made it possible to identify

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remarkable connections. In particular, it pointed out the connection with string theory, where such Riemann surfaces have a concrete physical meaning. Geometrical engineering [7] as well as M-theory/type-IIA methods [8, 9, 10], where a fivebrane is wrapped over a Riemann surface such that the theory living in the flat four dimensional part of the fivebrane become a four dimensional gauge theory, have greatly enlarged the Seiberg-Witten curves that can be found. Therefore, those connections have also enlarged the solutions of $\mathcal{N}=2$ gauge theories that can be studied.

Once the appropriate Riemann surface or algebraic curve is found for a given theory the order parameter of the theory, a_i, and its duals, a_D^i, are defined as the period integrals of the Seiberg-Witten differential defined over the Riemann surface. This is done in such a way that the prepotential of the theory is implicitly defined as $a_D^i = \partial\mathcal{F}/\partial a_i$. Then, once one finds the appropriate Riemann surface or algebraic curve for a given theory, the goal is to compute the period integrals and integrate them to find the prepotential $\mathcal{F}(a)$.

For classical groups, with gauge multiplet and N_f hypermultiplets in the fundamental representation the Seiberg-Witten curves encoding the solution of the theory are all hyperelliptic [2]–[6], (*i.e.* those curves have the form $y^2+P_1(x)y+P_2(x)=0$). Nevertheless, for example for $\mathrm{SU}(N_1)\times\mathrm{SU}(N_2)$ or $SU(N)$ with matter in the symmetric or antisymmetric representation, the appropriate curves are non-hyperelliptic, but are cubic [8, 9, 10] (i.e. of the form $y^3+P_1(x)y^2+P_2(x)y+P_3(x)=0$). Therefore, the problem that appears to compute the prepotential of the theory is how to evaluate the period integrals for non-hyperelliptic curves. A considerable effort has been done in this direction using a perturbation expansion of the non-hyperelliptic curve around its hyperelliptic approximation [11]–[14]. Nevertheless, the computation of the dual periods using that method gets very complicated and just allows one to compute the first instanton correction to the prepotential.

In that sense, it is always useful to find a method that let us determine the form of $\mathcal{F}$ without going through the actual computation of the periods. For $\mathcal{N}=2$ theories with classical gauge groups and matter in the fundamental representation such methods were developed in [15, 16], expressing the logarithmic derivative of the prepotential with respect to the quantum scale of the gauge theory in terms of the moduli of the curve [15, 17, 18, 19]. Also similar methods to compute recursively the instanton corrections to the prepotential were developed in [20, 21] using the connections of $\mathcal{N}=2$ theories with integrable systems. Inspired by this fact, in this paper we develop a method to compute the instanton correction to the prepotential recursively without computing the dual periods. In particular, we find the logarithmic derivatives of the prepotential for the non-hyperelliptic curves under study and use them to compute the prepotential avoiding the actual computation of the dual periods of the Seiberg-Witten differential. This method is the non-hyperelliptic generalization of the work done in [15, 16, 22] for hyperelliptic curves, and substantially simplifies the calculations done in [11]–[14]. Furthermore, this simplification allows us to compute recursively the instanton corrections to the prepotential in a remarkably straightforward way.

The structure of the paper is as follows: In the next section we review the form of the Seiberg-Witten curves for $\mathcal{N}=2$ supersymmetric gauge theory with gauge group $\mathrm{SU}(N_1)\times\mathrm{SU}(N_2)$, that are derived from M-theory considerations. We also study the form of those curves and analyse the information that we can extract from them. In section 3 we calculate the logarithmic derivatives of the prepotential with respect to the quantum scales of both groups in terms of the moduli of the curve, using the Riemann bilinear relations. In section 4 we develop a method to calculate the instanton corrections to the prepotential recursively using the previously calculated equation. We also extend, in section 5, the results of the previous section to the non-hyperelliptic curves obtained for SU(N) theories with matter in the symmetric and antisymmetric representation. Finally, in section 6 we present the conclusions.

2. $\mathcal{N} = 2$ supersymmetric SU(N_1) $\times$ SU(N_2) Yang-Mills theories

In this paper we will focus in a $\mathcal{N} = 2$ supersymmetric SU(N_1) $\times$ SU(N_2) theory with one massless hypermultiplet in the $(N_1, \bar{N}_2)$ bifundamental representation, together with N_{f_1} and N_{f_2} matter hypermultiplets in the fundamental representation of SU(N_1) and SU(N_2) respectively. This theory has a chiral multiplet in the adjoint representation of SU(N_1) that contains a complex scalar field ϕ and a chiral multiplet in the adjoint representation of SU(N_2) that contains a complex scalar field $\hat{\phi}$. This theory has a classical potential with flat directions that parametrizes the classical moduli space of the theory. Along such flat directions $[\phi, \bar{\phi}]$ and $[\hat{\phi}, \bar{\hat{\phi}}]$ vanish, and the symmetry is broken to $\mathrm{U}(1)^{N_1-1} \times \mathrm{U}(1)^{N_2-1}$. The low energy solution of the theory is encoded in a particular Riemann surface that allows us to compute the prepotential of the theory, this surface being derived from M-theory considerations.

2.1 Seiberg-Witten curves

The curve for this theory was derived by Witten in [8] by considering, in type-IIA string theory, D4-branes stretched between NS fivebranes (see figure 1). In this context, the Seiberg-Witten curve appears when this configuration is lifted to M-theory, as the configuration becomes a single fivebrane wrapped over a Riemann surface. This Riemann surface

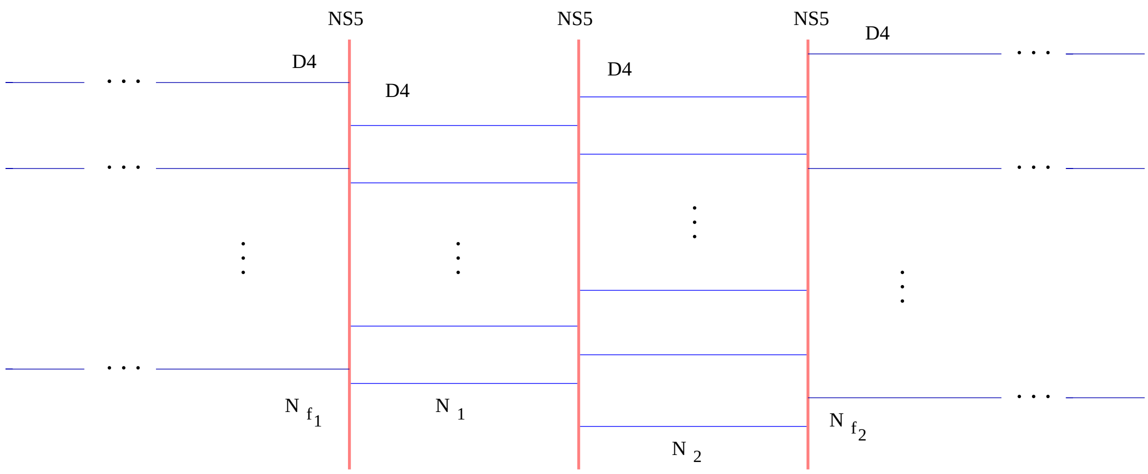

Figure 1: Brane picture of SU(N_1) $\times$ SU(N_2) gauge theories.

is, in fact, the Seiberg-Witten curve of the $\mathcal{N}=2$ theory. For the theory under study the Seiberg-Witten curve is given by [8]

$$P_0(x)\,y^3 - P_1(x)\,y^2 + \Lambda_1^{\beta_1} P_2(x)\,y - \Lambda_1^{2\beta_1}\Lambda_2^{\beta_2}\,P_3(x) = 0\,, \tag{2.1}$$

where the coefficients β_1 and β_2 are given by

$$\beta_1 = 2N_1 - N_{f_2} - N_1\,, \qquad \beta_2 = 2N_2 - N_{f_1} - N_2\,, \tag{2.2}$$

and Λ_1, Λ_2 denote the quantum scales of the two gauge groups. The requirement of asymptotic freedom, and restriction to the Coulomb phase, implies that Λ_1 and Λ_2 appear with positive powers in (2.1), that is, $\beta_1,\,\beta_2 > 0$.

Also in (2.1) the polynomials $P_1(x)$ and $P_2(x)$ denote the characteristic polynomial of $\mathrm{SU}(N_1)$ and $\mathrm{SU}(N_2)$ respectively, and are given by

$$P_1(x) = \prod_{i=1}^{N_1}(x-e_i) = x^{N_1} - \sum_{k=2}^{N_1} u_k x^{N_1-k} \tag{2.3}$$

$$P_2(x) = \prod_{i=1}^{N_2}(x-\hat{e}_i) = x^{N_2} - \sum_{k=2}^{N_2} \hat{u}_k x^{N_2-k}\,, \tag{2.4}$$

and the polynomials $P_0(x)$, and $P_3(x)$ depend just on the mass of the hypermultiplets m_f, $\hat{m}_f$, and are given by

$$P_0(x) = \prod_{i=1}^{N_{f_1}}(x+m_i) = x^{N_{f_1}} + \sum_{k=1}^{N_{f_1}} t_k x^{N_{f_1}-k} \tag{2.5}$$

$$P_3(x) = \prod_{i=1}^{N_{f_2}}(x+\hat{m}_i) = x^{N_{f_2}} + \sum_{k=1}^{N_{f_2}} \hat{t}_k x^{N_{f_2}-k}\,. \tag{2.6}$$

The $(N_1-1)+(N_2-1)$ dimensional moduli space is parametrized classically by e_i $(1\le i\le N_1)$ and $\hat{e}_i$ $(1\le i\le N_2)$, which are the eigenvalues of ϕ and $\hat{\phi}$ respectively, and satisfy the constraints $\sum_{i=1}^{N_1} e_i = 0$ and $\sum_{i=1}^{N_2}\hat{e}_i = 0$. Nevertheless, one should keep in mind that neither of these parameters are invariant under Weyl transformations. On the contrary the symmetric polynomials u_k, $\hat{u}_k$ in (2.3), (2.4), provide faithful coordinates for the moduli space of vacua, so all the physical quantities extracted from the curve must be given in terms of those polynomials.

One interesting feature that this Seiberg-Witten curve (2.1) presents is that the map

$$y \to \frac{\Lambda_1^{\beta_1}\Lambda_2^{\beta_2}}{y} \tag{2.7}$$

interchanges the gauge groups, that is, $\mathrm{SU}(N_1)\leftrightarrow\mathrm{SU}(N_2)$. This characteristic happens to be very useful for the calculations we will describe in the following sections.

The curve (2.1) is a non-hyperelliptic cubic curve with $2N_1+2N_2$ branch points that we will denote $e_i^{\pm}$, $i=1,\dots,N_1$ and $\hat{e}_j^{\pm}$, $j=1,\dots,N_2$. For small Λ_1 and Λ_2, those branch cuts are perturbations of e_i and $\hat{e}_i$ respectively. We view then the Riemann surface as a three-fold branched covering of the Riemann sphere, with branches one and two connected by N_1 square-root branch cuts joining e_i^+ and e_i^- and branches two and three connected by N_2 square-root branch cuts joining $\hat{e}_i^+$ and $\hat{e}_i^-$.

In general terms, a Riemann surface defined by the relation $F(x,y) = \sum_{i,j} a_{ij} y^i x^j = 0$ has a set of holomorphic differentials given by $d\omega_{i,j} = \frac{y^{i-1}x^j}{\partial_x F} dy$, each one of them associated with each one of the moduli a_{ij} of the curve. In our case, we will have $g = N_1 + N_2 - 2$ holomorphic differentials associated with each one of the moduli u_k, $\hat{u}_k$ [23]. For the parametrization of the curve (2.1) we find the following set of holomorphic differentials

$$d\omega_k = -\frac{y x^{N_1-k}}{\partial_x F} dy \,, \qquad d\hat{\omega}_k = \frac{\Lambda_1^{\beta_1} x^{N_2-k}}{\partial_x F} dy \,. \tag{2.8}$$

Note that (2.7), that interchanges the role of the gauge groups in the Seiberg-Witten curve also interchanges the set holomorphic differentials $d\omega_k \leftrightarrow d\hat{\omega}_k$, as it should.

The SW differential for these curves is

$$dS_{SW} = x \frac{\mathrm{d}y}{y} \,, \tag{2.9}$$

which takes a different value on each one of the three Riemann branches. This differential is defined in such a way that its derivative with respect to the moduli of the curve shall give the holomorphic differentials (2.8). Note that this condition is fulfilled in our case as $d\omega_k = \partial_{u_k} dS_{SW}$ and $d\hat{\omega}_k = \partial_{\hat{u}_k} dS_{SW}$.

2.2 Analysis of the curves

Working with non-hyperelliptic curves, even with cubic ones, is not an easy task. The main reason for this is that one needs to know the algebraic solution of the curve (that is, $y = y(x)$) in order to be able to compute the periods of the Seiberg-Witten differential. Solutions are generically not known, unless for cubic curves. But, even in the cubic case, they are too complicated to be useful. Nevertheless, those solutions can be simplified in the following way: If we take the limit $\Lambda_2 \to 0$ on (2.1) we get

$$P_0(x) y^2 - P_1(x) y + \Lambda_1^{\beta_1} P_2(x) = 0 \,, \tag{2.10}$$

that is, we recover the hyperelliptic curve that describes the case of a supersymmetric Yang-Mills theory with gauge group $\mathrm{SU}(N_1)$ with $N_{f_1} + N_2$ matter hypermultiplets in the fundamental representation of the gauge group.

This can also be seen from the brane picture, as shown in figure 2. Taking the quantum scale Λ_2 to zero is the same as taking the distance between the second and the third $NS5$-branes in figure 1 to infinity. The gauge coupling of the $\mathrm{SU}(N_2)$ theory then will go to zero and the moduli of the theory are seen just as constants with respect to the quantum scale Λ_1, that is, they are seen as matter in the fundamental representation of $\mathrm{SU}(N_1)$. Since the curve (2.1) obeys the involution (2.7), the same is true when $\Lambda_1 \to 0$ but interchanging the role of the gauge groups. Therefore, when $\Lambda_1 \to 0$, we would have a $\mathrm{SU}(N_2)$ gauge theory with $N_{f_2} + N_1$ matter hypermultiplets in the fundamental representation of $\mathrm{SU}(N_2)$.

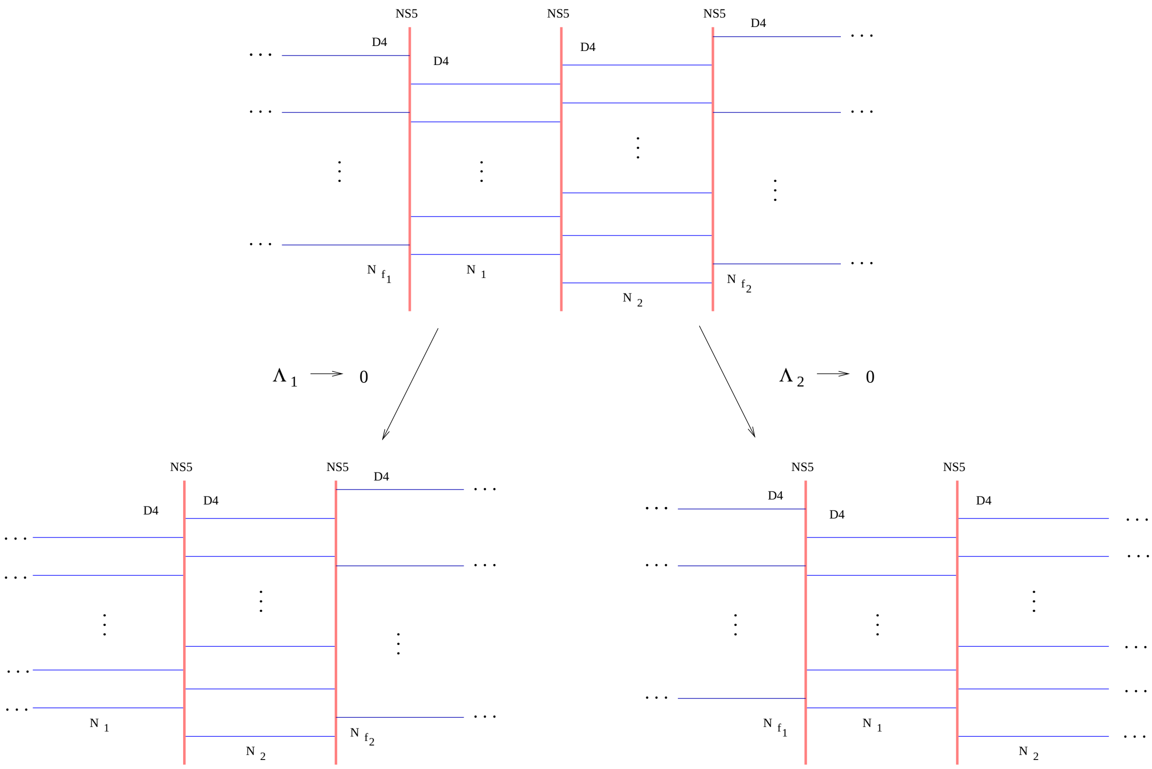

Figure 2: Different limits of the theory.

Then, it is clear that one can obtain the solutions to the non-hyperelliptic curve (2.1) by means of a systematic series expansion of the curve around $\Lambda_2^{\beta_2} = 0$, with the zeroth-order term being a hyperelliptic curve. This idea was exploited in [11]–[14]. Using this, the solutions to (2.1) accurate to the order $\mathcal{O}(\Lambda_2^{2\beta_2})$, are

$$\begin{aligned} y_1 &= \sum_{k=0}^{\infty} y_k^{(1)} \Lambda_2^{k\beta_2} = \frac{P_1 + r}{2P_0} + \frac{\Lambda_1^{\beta_1} P_3 (P_1 - r)}{2P_2 r} \Lambda_2^{\beta_2} + \\ &\quad + \frac{\Lambda_1^{\beta_1} P_3^2 (P_1^4 + 6\Lambda_1^{2\beta_1} P_0^2 P_2^2 - 6\Lambda_1^{\beta_1} P_0 P_2 P_1^2 - P_1 r^3)}{2P_2^3 r^3} \Lambda_2^{2\beta_2} + \cdots , \end{aligned} \tag{2.11}$$

$$\begin{aligned} y_2 &= \sum_{k=0}^{\infty} y_k^{(2)} \Lambda_2^{k\beta_2} = \frac{P_1 - r}{2P_0} - \frac{\Lambda_1^{\beta_1} P_3 (P_1 + r)}{2P_2 r} \Lambda_2^{\beta_2} - \\ &\quad - \frac{\Lambda_1^{\beta_1} P_3^2 (P_1^4 + 6\Lambda_1^{2\beta_1} P_0^2 P_2^2 - 6\Lambda_1^{\beta_1} P_0 P_2 P_1^2 + P_1 r^3)}{2P_2^3 r^3} \Lambda_2^{2\beta_2} + \cdots , \end{aligned} \tag{2.12}$$

$$y_3 = \sum_{k=1}^{\infty} y_k^{(3)} \Lambda_2^{k\beta_2} = \frac{\Lambda_1^{\beta_1} P_3}{P_2} \Lambda_2^{\beta_2} + \frac{\Lambda_1^{\beta_1} P_1 P_3^2}{P_2^3} \Lambda_2^{2\beta_2} + \cdots . \tag{2.13}$$

where $r \equiv \sqrt{P_1^2 - 4\Lambda_1^{\beta_1} P_0 P_2}$, and $y_0^{(1)}$ and $y_0^{(2)}$ are the solutions to the curve (2.10), that is, the hyperelliptic solutions (in the limit $\Lambda_2 = 0$). Notice that the involution map (2.7) interchanges the branches $y_1 \leftrightarrow y_3$.

The perturbative expansion in $\Lambda_2^{\beta_2}$, (2.11)–(2.13), induces a comparable expansion for the SW differential (2.9). For example, in the branch one, using (2.11), we have

$$dS_{SW}^{(1)} = \left(dS_{SW}^{(1)}\right)_I + \left(dS_{SW}^{(1)}\right)_{II}\Lambda_2^{\beta_2} + \left(dS_{SW}^{(1)}\right)_{III}\Lambda_2^{2\beta_2} + \cdots \tag{2.14}$$

where

$$\left(dS_{SW}^{(1)}\right)_I = x\frac{dy_0^{(1)}}{y_0^{(1)}} = x\frac{\partial_x\left(\frac{P_1}{(P_0P_2)^{1/2}}\right)}{\sqrt{\frac{P_1^2}{P_2P_0} - 4\Lambda_1^{\beta_1}}}dx + \frac{1}{2}x\left(\frac{\partial_x P_2}{P_2} - \frac{\partial_x P_0}{P_0}\right)dx\,, \tag{2.15}$$

Note that this expression is the usual one for the SW differential for a hyperelliptic curve, up to terms (the last term in (2.15)) that do not contribute to the periods defined in the branch one. The subsequent corrections are

$$\left(dS_{SW}^{(1)}\right)_{II} = -\frac{y_1^{(1)}}{y_0^{(1)}}dx = -\frac{\Lambda_1^{\beta_1}P_3P_0(P_1-r)}{P_2r(P_1+r)}dx\,. \tag{2.16}$$

$$\begin{aligned}\left(dS_{SW}^{(1)}\right)_{III} &= -\left(\frac{y_2^{(1)}}{y_0^{(1)}} + \frac{(y_1^{(1)})^2}{2y^{(1)}}\right)dx\\ &= \frac{2\Lambda_1^{2\beta_1}P_3^2P_0^2((9P_1-5r)\Lambda_1^{\beta_1}P_0P_2 - 2P_1^2(P_1-r))}{P_2^2r^3(P_1+r)^2}dx\,.\end{aligned} \tag{2.17}$$

Equation (2.7) maps the branches as follows: $y_1 \leftrightarrow y_3$ and $y_2 \leftrightarrow y_2$. Using $y_3 = \Lambda_1^{\beta_1}\Lambda_2^{\beta_2}/y_1$, we may express the expansion for $dS_{SW}^{(3)}$ in terms of a comparable one for $dS_{SW}^{(1)}$, for which $\mathrm{SU}(N_1) \leftrightarrow \mathrm{SU}(N_2)$, with the approximation (2.13) exhibiting the branch cuts which connect branches 2 and 3.

Given the SW differential to the required accuracy, we are able to compute the order parameters and dual order parameters to the appropriate order on both quantum scales, as they are given in terms of period integrals of the Seiberg-Witten differential. Nevertheless, as we pointed out earlier, we will avoid the computation of the dual periods using the derivatives of the prepotential.

3. Derivatives of the prepotential

In $\mathrm{SU}(N)$ gauge theories with matter in the fundamental representation for which the Seiberg-Witten curve is a hyperelliptic one, the computation of the logarithmic derivative of the prepotential with respect to the quantum scale in terms of parameters of the curve [15, 17, 18, 19] is found to be very useful to calculate the instanton corrections to the prepotential [15, 16]. The reason is that one does not need to compute the dual periods to obtain them so the calculation is much simpler. In this section we will find a renormalization group type equation that will help us to calculate the instanton corrections to the prepotential.

We will do this with the help of the Riemann bilinear relation, along the lines of what was done in $\mathrm{SU}(N)$ gauge theories [18, 19]. The first thing we have to consider is that the effective (field dependent, dimensionless) gauge coupling is given by the second derivative of the prepotential. $\mathcal{F}$ is thus a homogeneous function of weight two on the variables A_i, $(i = 1, \ldots, g)$, M_j, $(j = 1, \ldots, N_{f_1} + N_{f_2})$, and on the quantum scales of both groups Λ_1, Λ_2. Therefore satisfies the Euler equation

$$2\mathcal{F} = \left(\Lambda_1 \frac{\partial}{\partial \Lambda_1} + \Lambda_2 \frac{\partial}{\partial \Lambda_2} + \sum_{i=1}^{g} A_i \frac{\partial}{\partial A_i} + \sum_{j=1}^{N_{f_1}+N_{f_2}} M_j \frac{\partial}{\partial M_j} \right) \mathcal{F} \, . \tag{3.1}$$

Taking the derivatives with respect to the moduli of the curve u_k, and also using the definition of $A_D^i = \frac{\partial \mathcal{F}}{\partial A_i}$ one obtains

$$\frac{\partial}{\partial u_k} \left(\Lambda_1 \frac{\partial}{\partial \Lambda_1} + \Lambda_2 \frac{\partial}{\partial \Lambda_2} + \sum_j M_j \frac{\partial}{\partial M_j} \right) \mathcal{F} = \sum_i \left(A_D^i \frac{\partial}{\partial u_k} A_i - A_i \frac{\partial}{\partial u_k} A_D^i \right) . \tag{3.2}$$

Using now the definitions of A_i, A_D^i as the periods of the Seiberg-Witten differential we arrive at

$$\frac{\partial}{\partial u_k} \left(\Lambda_1 \frac{\partial}{\partial \Lambda_1} + \Lambda_2 \frac{\partial}{\partial \Lambda_2} + \sum_j M_j \frac{\partial}{\partial M_j} \right) \mathcal{F} = \sum_{i=1}^{g} \oint_{\alpha_i} d\omega_k \oint_{\beta^i} dS_{SW} - \oint_{\beta^i} d\omega_k \oint_{\alpha_i} dS_{SW} \, , \tag{3.3}$$

where we have used $\partial_{u_k} dS_{SW} = d\omega_k$.

The right hand side of this equation can be evaluated with the help of a Riemann bilinear relation [24]. In a general case, if we have two abelian differentials $d\Omega$ and $d\Omega'$ defined on a genus g Riemann surface, the Riemann bilinear relations read

$$\sum_{i=1}^{g} \oint_{\alpha_i} d\Omega \oint_{\beta^i} d\Omega' - \oint_{\beta^i} d\Omega \oint_{\alpha_i} d\Omega' = \frac{1}{2\pi i} \sum_{s_\lambda} \mathrm{res}_{s_\lambda} (\Omega d\Omega') \tag{3.4}$$

where, when $d\Omega$ is a holomorphic differential, s_λ denote the poles of $d\Omega'$. In this case we have $d\Omega = d\omega_k$ and $d\Omega' = dS_{SW}$. As can be seen from (2.9) the Seiberg-Witten differential dS_{SW} is an abelian differential of the third kind with poles at $x \to \infty$ and $x \to m_j, \hat{m}_j$, and also $d\omega_k$ is the holomorphic differential (2.8). Then $s_\lambda = \{\infty, m_j, \hat{m}_j\}$. As our Riemann surface is a three branched cover of the sphere, we have that

$$d\omega_k \stackrel{x\to\infty}{\longrightarrow} \begin{cases} x^{-k} dx + \mathcal{O}(x^{-k-1}) & \text{branch 1} \\ -x^{-k} dx + \mathcal{O}(x^{-k-1}) & \text{branch 2} \\ \mathcal{O}(x^{-\beta_2 - k}) & \text{branch 3} \end{cases} , \tag{3.5}$$

and

$$dS_{SW} \stackrel{x\to\infty}{\longrightarrow} \begin{cases} (N_1 - N_{f_1}) dx + \mathcal{O}(x^{-1}) & \text{branch 1} \\ -(N_1 - N_2) dx + \mathcal{O}(x^{-1}) & \text{branch 2} \\ \mathcal{O}(1) & \text{branch 3} \end{cases} . \tag{3.6}$$

Also when $x \to m_j$, $\hat{m}_j$

$$dS_{SW} \stackrel{x\to m_j}{\longrightarrow} \begin{cases} -\frac{m_j}{x-m_j} dx + \cdots \\ \cdots \\ \cdots \end{cases} , \qquad dS_{SW} \stackrel{x\to \hat{m}_j}{\longrightarrow} \begin{cases} \cdots & \text{branch 1} \\ \frac{\hat{m}_j}{x-\hat{m}_j} dx + \cdots & \text{branch 2} \\ \cdots & \text{branch 3} \end{cases} , \tag{3.7}$$

where by $\cdots$ we denote terms that are regular when $x \to m_j,\, \hat{m}_j$. Now, using (3.4)–(3.7) we can write eq. (3.3) as

$$\frac{\partial}{\partial u_k}\left(\Lambda_1 \frac{\partial}{\partial \Lambda_1} + \Lambda_2 \frac{\partial}{\partial \Lambda_2} + \sum_j M_j \frac{\partial}{\partial M_j}\right)\mathcal{F} = \frac{\beta_1}{2\pi i}\delta_{k,2} - \frac{1}{2\pi i}\sum_j M_j\, \omega_k\, |_{x=M_j} + \\ + \frac{1}{2\pi i}\sum_j M_j\, \omega_k\, |_{x=\infty} \\ = \frac{\beta_1}{2\pi i}\delta_{k,2} + \frac{1}{2\pi i}\sum_j M_j \int_{M_j}^{\infty} d\omega_k \,. \tag{3.8}$$

We now want to further simplify this formula. Using $\frac{\partial \mathcal{F}}{\partial u_k} = \sum_i \frac{\partial A_i}{\partial u_k} A_D^i$ we get to

$$\sum_j M_j \frac{\partial \mathcal{F}}{\partial M_j \partial u_k} = \sum_{i,j} M_j \frac{\partial A_i}{\partial u_k}\frac{\partial A_D^i}{\partial M_j} = \sum_{i,j} M_j \oint_{\alpha_i} d\omega_k \oint_{\beta^i} \frac{\partial dS_{SW}}{\partial M_j}\,, \tag{3.9}$$

so now we can use again the Riemann bilinear relation and we arrive to

$$\sum_{i,j} M_j \oint_{\alpha_i} d\omega_k \oint_{\beta^i} \frac{\partial dS_{SW}}{\partial M_j} = \sum_j M_j \sum_{s_\lambda} \mathrm{res}_{s_\lambda} \omega_k \frac{\partial dS_{SW}}{\partial M_j}\,, \tag{3.10}$$

as $\oint_{\alpha_i} \frac{\partial dS_{SW}}{\partial M_j} = \partial A_i/\partial M_j = 0$. Furthermore, as can be read off from (3.6), (3.7), the differential $\partial dS_{SW}/\partial M_j$ is a third kind differential with a pole of order 1 in $x = \infty$, M_j with residue $+1$ and -1 respectively. Therefore

$$\sum_{s_\lambda} \mathrm{res}_{s_\lambda} \omega_k \frac{\partial dS_{SW}}{\partial M_j} = \omega_k\, |_\infty - \omega_k\, |_{M_j} = \int_{M_j}^{\infty} d\omega_k \,. \tag{3.11}$$

So we finally arrive to the expression

$$\frac{\partial}{\partial u_k}\left(\Lambda_1 \frac{\partial}{\partial \Lambda_1} + \Lambda_2 \frac{\partial}{\partial \Lambda_2}\right)\mathcal{F} = \frac{\beta_1}{2\pi i}\delta_{k,2}\,. \tag{3.12}$$

Then, integrating eq. (3.12) with respect to u_k we get

$$\left(\frac{\partial}{\partial \log \Lambda_1} + \frac{\partial}{\partial \log \Lambda_2}\right)\mathcal{F} = \frac{\beta_1}{2\pi i} u_2 + \text{terms indep. of}\, u_k\,. \tag{3.13}$$

We can actually perform a similar calculation but taking the derivative of (3.1) with respect to the moduli $\hat{u}_k$, and we will get to

$$\left(\frac{\partial}{\partial \log \Lambda_1} + \frac{\partial}{\partial \log \Lambda_2}\right)\mathcal{F} = \frac{\beta_2}{2\pi i} \hat{u}_2 + \text{ terms indep. of}\, \hat{u}_k\,. \tag{3.14}$$

Therefore, from (3.13) and (3.14), we can conclude that

$$\left(\frac{\partial}{\partial \log \Lambda_1} + \frac{\partial}{\partial \log \Lambda_2}\right)\mathcal{F} = \frac{1}{2\pi i}\left(\beta_1 u_2 + \beta_2 \hat{u}_2\right) + \text{terms indep. of}\, u_k\,, \hat{u}_k \tag{3.15}$$
$$= \frac{1}{2\pi i}\left(\frac{\beta_1}{2}\sum_{i=1}^{N_1} e_i^2 + \frac{\beta_2}{2}\sum_{i=1}^{N_2} \hat{e}_i^2\right) + \text{terms indep. of}\, e_k\,, \hat{e}_k\,,$$

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where we have used $2u_2 = \sum_{i=1}^{N_1} e_i^2$, $2\hat{u}_2 = \sum_{i=1}^{N_2} \hat{e}_i^2$. The terms independent of u_k, $\hat{u}_k$, are unphysical constant terms that depend on the exact parametrization of the prepotential.

It is interesting to point out that the final result appears to be just the sum of the equations for $\mathrm{SU}(N_1)$ and $\mathrm{SU}(N_2)$. Nevertheless that is not the case, as u_2 depends both on a_i and $\hat{a}_i$, and the same happens for $\hat{u}_2$. That means that there is a non-trivial mixing between both groups, as is expected in the presence of a hypermultiplet in the bifundamental representation.

4. Instanton corrections to the prepotential

The quantum relations between the low-energy coordinates of the moduli space a_i, $\hat{a}_i$, a_D^j, $\hat{a}_D^j$, and the parameters on the curve, are implicitly given by the period integrals

$$a_k = \oint_{\alpha_k} dS_{SW}^{(1)}\,, \qquad \hat{a}_k = \oint_{\alpha_{N_1+k}} dS_{SW}^{(3)}\,, \tag{4.1}$$

and

$$a_D^k = \oint_{\beta^k} dS_{SW}^{(1)}\,, \qquad \hat{a}_D^k = \oint_{\beta^{N_1+k}} dS_{SW}^{(3)}\,, \tag{4.2}$$

where (α_i, β^j) constitute a symplectic basis of homology cycles of the Riemann surface with canonical intersections. The cycle α_k is chosen to be a simple contour enclosing the branch cut connecting e_k^+ with e_k^- $(k = 1, \ldots, N_1)$ on branch 1, while for $k = N_1 + 1, \ldots, N_1 + N_2$ similarly encloses the branch cut connecting $\hat{e}_k^+$ with $\hat{e}_k^-$ on branch 3. Then we have that $\sum_k a_k = 0$ and $\sum_k \hat{a}_k = 0$. As a function of these variables, the canonical prepotential $\mathcal{F}$ is implicitly defined by the equations

$$a_D^k(a_j, \hat{a}_j) = \frac{\partial \mathcal{F}}{\partial a_k} - \frac{\partial \mathcal{F}}{\partial a_{N_1}}\,, \qquad k = 1, \ldots, N_1 - 1\,, \tag{4.3}$$

$$\hat{a}_D^k(a_j, \hat{a}_j) = \frac{\partial \mathcal{F}}{\partial \hat{a}_k} - \frac{\partial \mathcal{F}}{\partial \hat{a}_{N_2}}\,, \qquad k = 1, \ldots, N_2 - 1\,, \tag{4.4}$$

where we have taken into account the constraint $\sum_k a_k = \sum_k \hat{a}_k = 0$. The exact determination of the prepotential, as is shown in (4.3), (4.4), involves the integration of the functions $a_D^k(a_j, \hat{a}_j)$, $\hat{a}_D^k(a_j, \hat{a}_j)$ for which there is no closed form available. In this context the existence of an algorithm that enables us to determine the exact form of $\mathcal{F}$ without going through the actual computation of the dual periods is welcome.

The holomorphic prepotential can be expressed as

$$\mathcal{F} = \mathcal{F}_{\text{classic}} + \mathcal{F}_{\text{1-loop}} + \mathcal{F}_{\text{instanton}}\,, \tag{4.5}$$

since the perturbative corrections saturate at one-loop in $\mathcal{N} = 2$ theories [25] but there is an infinite series of non-perturbative instanton contributions. The one-loop perturbative correction to the prepotential has been calculated in [13] for the case of product gauge groups by calculating the period integrals using the hyperelliptic approximation, and is given by

$$\mathcal{F}_{\text{1-loop}} = \frac{i}{4\pi}\Bigg\{\sum_{\substack{i,j=1\\ i<j}}^{N_1}(a_i-a_j)^2\log(a_i-a_j)^2+\sum_{\substack{\alpha,\beta=1\\ \alpha<\beta}}^{N_2}(\hat{a}_\alpha-\hat{a}_\beta)^2\log(\hat{a}_\alpha-\hat{a}_\beta)^2-$$
$$-\frac{1}{2}\sum_{i=1}^{N_1}\sum_{\alpha=1}^{N_2}(a_i-\hat{a}_\alpha)^2\log(a_i-\hat{a}_\alpha)^2-\frac{1}{2}\sum_{i=1}^{N_1}\sum_{f=1}^{N_{f_1}}(a_i-m_f)^2\log(a_i-m_f)^2-$$
$$-\frac{1}{2}\sum_{\alpha=1}^{N_2}\sum_{f=1}^{N_{f_2}}(\hat{a}_\alpha-\hat{m}_f)^2\log(\hat{a}_\alpha-\hat{m}_f)^2\Bigg\}. \tag{4.6}$$

For the non-perturbative corrections in (4.5), note that the n-th order instanton corrections to the prepotential is not just the sum of contributions proportional to $\Lambda_1^{n\beta_1}$, $\Lambda_2^{n\beta_2}$, but all possible combinations of terms $\Lambda_1^{n_1\beta_1}\Lambda_2^{n_2\beta_2}$ with $n_1+n_2=n$. Therefore we will take the form of $\mathcal{F}_{\text{instanton}}$ to be

$$\mathcal{F}_{\text{instanton}}(a_i,\hat{a}_i)=\frac{1}{2\pi i}\sum_{k,l=1}^{\infty}\mathcal{F}_{k,l}(a_i,\hat{a}_i)\Lambda_1^{k\beta_1}\Lambda_2^{l\beta_2}\,. \tag{4.7}$$

We will check this assumption in the next subsection.

4.1 Computation of instanton corrections

To compute the instanton corrections to the prepotential we need to perform the period integral (4.1) for the order parameters. Those periods can be computed by reducing the evaluation of the integrals to a set of residue calculations in the same way as it was first done in [22]. The argument is the following: We can always consider the distance between the contour α_k and the branch cut connecting e_k^+ ($\hat{e}_k^+$) with e_k^- ($\hat{e}_k^-$) to be much larger than $\Lambda_1^{\beta_1}$, $\Lambda_2^{\beta_2}$. Then, if we expand dS_{SW} in power series around $\Lambda_1=0$ and $\Lambda_2=0$, the integrals (4.1) over the cycle α_k can be performed just by calculating residues at the points e_k (for α_k, $k=1,\dots,N_1$), or $\hat{e}_k$ (for α_k, $k=N_1+1,\dots,N_1+N_2$).

Therefore, if we want to compute in this way the order parameters a_i we need to expand $dS_{SW}^{(1)}$ around $\Lambda_1=0$, $\Lambda_2=0$. Using (2.15)–(2.17) we get

$$\left(dS_{SW}^{(1)}\right)_I \simeq x\frac{\partial_x P_1}{P_1}+\sum_{k=1}^{\infty}\frac{(2k-1)!}{(k!)^2}\left(\frac{P_0P_2}{P_1^2}\right)^k\Lambda_1^{k\beta_1}\,, \tag{4.8}$$
$$\left(dS_{SW}^{(1)}\right)_{II} \simeq -P_3\sum_{k=1}^{\infty}\frac{(2k)!}{(k-1)!(k+1)!}\frac{P_0^{k+1}P_2^{k-1}}{P_1^{2k+1}}\Lambda_1^{(k+1)\beta_1}\Lambda_2^{\beta_2}\,, \tag{4.9}$$
$$\vdots$$

where by $\simeq$ we denote terms up to total derivatives or terms that have no residue at the corresponding cycles.

Using this we can write the order parameters a_i as

$$a_i=\text{res}_{e_i}dS_{SW}^{(1)}=e_i+\sum_{k=1}^{\infty}\Delta_i^{(k)}(e_i)\Lambda_1^{\beta_1k}+\sum_{k,l=1}^{\infty}\Delta_i^{(k,l)}(e_i)\Lambda_1^{\beta_1(k+2l-1)}\Lambda_2^{\beta_2l}\,, \tag{4.10}$$

where the exact form of the functions $\Delta_i^{(k)}$, $\hat{\Delta}_i^{(k)}$, $\Delta_i^{(k,l)}$, $\hat{\Delta}_i^{(k,l)}$ is obtained using (4.8), (4.9), and is given by

$$\Delta_i^{(k)}(x) = \frac{1}{(k!)^2}\left(\frac{\partial}{\partial x}\right)^{2k-1}\left(\frac{P_0(x)^k P_2(x)^k}{\prod_{j\neq i}(x-e_j)^{2k}}\right), \tag{4.11}$$

$$\Delta_i^{(k,1)}(x) = -\frac{k}{k!(k+1)!}\left(\frac{\partial}{\partial x}\right)^{2k}\left(P_3(x)\frac{(P_0(x))^{k+1}(P_2(x))^{k-1}}{\prod_{j\neq i}(x-e_j)^{2k+1}}\right), \tag{4.12}$$

$$\vdots$$

Also by a similar computation we obtain

$$\hat{a}_i = \hat{e}_i + \sum_{k=1}^{\infty}\hat{\Delta}_i^{(k)}(\hat{e}_i)\Lambda_2^{\beta_2 k} + \sum_{k,l=1}^{\infty}\hat{\Delta}_i^{(k,l)}(\hat{e}_i)\Lambda_1^{\beta_1 k}\Lambda_2^{\beta_2(l+2k-1)}, \tag{4.13}$$

where

$$\hat{\Delta}_i^{(k)}(x) = \frac{1}{(k!)^2}\left(\frac{\partial}{\partial x}\right)^{2k-1}\left(\frac{P_1(x)^k P_3(x)^k}{\prod_{j\neq i}(x-\hat{e}_j)^{2k}}\right), \tag{4.14}$$

$$\hat{\Delta}_i^{(k,1)}(x) = -\frac{k}{k!(k+1)!}\left(\frac{\partial}{\partial x}\right)^{2k}(P_0(x)\frac{(P_3(x))^{k+1}(P_1(x))^{k-1}}{\prod_{j\neq i}(x-\hat{e}_j)^{2k+1}}), \tag{4.15}$$

$$\vdots$$

Once we have the expression for the power expansion of the order parameters in Λ_1, Λ_2 one can use the equation (3.15) to compute the instanton correction to the prepotential. Just by inserting the expression for the prepotential (4.5) in eq. (3.15) one gets

$$\frac{\beta_1}{2}\sum_i a_i^2 + \frac{\beta_2}{2}\sum_i \hat{a}_i^2 + \sum_{k,l=1}^{\infty}(\beta_1 k+\beta_2 l)\mathcal{F}_{k,l}(a_j,\hat{a}_j)\Lambda_1^{\beta_1 k}\Lambda_2^{\beta_2 l} = \frac{\beta_1}{2}\sum_i e_i^2 + \frac{\beta_2}{2}\sum_i \hat{e}_i^2, \tag{4.16}$$

up to unphysical constant terms.

From here it is clear how to extract the instanton corrections to the prepotential from (4.16). The first thing we need is the expansions (4.10), (4.13) of the order parameters in terms of e_i, $\hat{e}_i$. Secondly, let us expand $\mathcal{F}_{k,l}(a_j,\hat{a}_j)$ in power series around $a_i = e_i$, and $\hat{a}_i = \hat{e}_i$. We have

$$\begin{aligned}\mathcal{F}_{k,l}(a_j,\hat{a}_j) &= \mathcal{F}_{k,l}(e_j,\hat{e}_j) + \sum_i \partial_{e_i}\mathcal{F}_{k,l}\Delta_i^{(1)}\Lambda_1^{\beta_1} + \sum_i \partial_{\hat{e}_i}\mathcal{F}_{k,l}\hat{\Delta}_i^{(1)}\Lambda_2^{\beta_2} \\ &+ \left(\sum_i \partial_{e_i}\mathcal{F}_{k,l}\Delta_i^{(2)} + +\frac{1}{2}\sum_{i_1,i_2}\partial_{e_{i_1}}\partial_{e_{i_2}}\mathcal{F}_{k,l}\Delta_{i_1}^{(1)}\Delta_{i_2}^{(1)}\right)\Lambda_1^{2\beta_1} + \\ &+ \left(\sum_i \partial_{\hat{e}_i}\mathcal{F}_{k,l}\hat{\Delta}_i^{(2)} + \frac{1}{2}\sum_{i_1,i_2}\partial_{\hat{e}_{i_1}}\partial_{\hat{e}_{i_2}}\mathcal{F}_{k,l}\hat{\Delta}_{i_1}^{(1)}\hat{\Delta}_{i_2}^{(1)}\right)\Lambda_2^{2\beta_2} + \\ &+ \sum_{i_1,i_2}\partial_{e_{i_1}}\partial_{\hat{e}_{i_2}}\mathcal{F}_{k,l}\Delta_{i_1}^{(1)}\hat{\Delta}_{i_2}^{(1)}\Lambda_1^{\beta_1}\Lambda_2^{\beta_2} + \ldots\end{aligned} \tag{4.17}$$

Now inserting (4.10), (4.13) and (4.17) into (4.16) it is possible to obtain the instanton corrections recursively just by identifying order by order in Λ_1, Λ_2 in eq. (4.16). For example for the first corrections we have[1]

$$
\begin{aligned}
\mathcal{F}_{1,0}(a_j,\hat{a}_j) &= -\sum_{i=1}^{N_1} a_i\Delta_i^{(1)}\,,\\
\mathcal{F}_{2,0}(a_j,\hat{a}_j) &= -\frac{1}{2}\sum_{i=1}^{N_1}\left(a_i\Delta_i^{(2)}+\frac{1}{2}\Delta_i^{(1)}\Delta_i^{(1)}+\partial_{a_i}\mathcal{F}_{1,0}\Delta_i^{(1)}\right),\\
\mathcal{F}_{1,1}(a_j,\hat{a}_j) &= -\frac{\beta_2}{\beta_1+\beta_2}\sum_{i=1}^{N_1}\partial_{a_i}\mathcal{F}_{0,1}\Delta_i^{(1)}-\frac{\beta_1}{\beta_1+\beta_2}\sum_{i=1}^{N_2}\partial_{\hat{a}_i}\mathcal{F}_{1,0}\hat{\Delta}_i^{(1)}\,,\\
\mathcal{F}_{3,0}(a_j,\hat{a}_j) &= -\frac{1}{3}\sum_{i=1}^{N_1}\Big(a_i\Delta_i^{(3)}+\Delta_i^{(1)}\Delta_i^{(2)}+2\partial_{a_i}\mathcal{F}_{2,0}\Delta_i^{(1)}+\partial_{a_i}\mathcal{F}_{1,0}\Delta_i^{(2)}+\\
&\qquad\qquad+\sum_{j=1}^{N_1}\frac{1}{2}\partial_{a_i}\partial_{a_j}\mathcal{F}_{1,0}\Delta_i^{(1)}\Delta_j^{(1)}\Big)\,,\\
\mathcal{F}_{2,1}(a_j,\hat{a}_j) &= -\frac{\beta_1}{2\beta_1+\beta_2}\sum_{i=1}^{N_1}\left(a_i\Delta_i^{(1,1)}+2\partial_{\hat{a}_i}\mathcal{F}_{2,0}\hat{\Delta}_i^{(1)}+\sum_{j=1}^{N_2}\partial_{a_i}\partial_{\hat{a}_j}\mathcal{F}_{1,0}\Delta_i^{(1)}\hat{\Delta}_j^{(1)}\right)-\\
&\quad-\frac{\beta_2}{2\beta_1+\beta_2}\sum_{i=1}^{N_1}\left(\partial_{a_i}\mathcal{F}_{0,1}\Delta_i^{(2)}+\frac{1}{2}\sum_{j=1}^{N_2}\partial_{a_i}\partial_{a_j}\mathcal{F}_{0,1}\Delta_i^{(1)}\Delta_j^{(1)}\right)-\\
&\quad-\frac{\beta_1+\beta_2}{2\beta_1+\beta_2}\sum_{i=1}^{N_1}\partial_{a_i}\mathcal{F}_{1,1}\Delta_i^{(1)}\,,\\
&\vdots
\end{aligned}
\tag{4.18}
$$

Note that the method developed here allow us to compute the instanton corrections to the prepotential recursively in a remarkably straightforward way up to arbitrary high orders. In fact, eq. (4.16) together with the hyperelliptic approximation $(dS_{SW})_I$ to the Seiberg-Witten differential, fixes the instanton corrections up to order two in both quantum scales (i.e.up to terms $\Lambda_1^{k\beta_1}\Lambda_2^{l\beta_2}$ with $k+l=2$). Also the next correction to the Seiberg-Witten differential $(dS_{SW})_{II}$ is enough to reach the fifth order instanton correction.

4.2 Examples

In this subsection we will write explicitly the instanton corrections obtained for some product gauge groups.

SU(2) × SU(2). For asymptotically free theories, the most general case that one can consider is the case with one matter hypermultiplet in the fundamental representation of

[1] We just list one half of the instanton corrections here. The other half are obtained from these ones just by using the fact that the flip $\mathrm{SU}(N_1)\leftrightarrow\mathrm{SU}(N_2)$ interchanges $\mathcal{F}_{k,l}\leftrightarrow\mathcal{F}_{l,k}$. Note that we only get contributions of the form (4.7) to the instanton corrections.

each group and one in the bifundamental. We obtain the following results

$$
\begin{aligned}
\mathcal{F}_{1,0} &= -\frac{m\hat{u}}{2u}\,, \\
\mathcal{F}_{0,1} &= -\frac{\hat{m}u}{2\hat{u}}\,, \\
\mathcal{F}_{1,1} &= \frac{m\hat{m}(u+\hat{u})}{4u\hat{u}}\,, \\
\mathcal{F}_{2,0} &= \frac{2u^2\hat{u}-3u\hat{u}^2+m^2(u^2+5\hat{u}^2-6u\hat{u})}{64u^3} \\
\mathcal{F}_{0,2} &= \frac{2u\hat{u}^2-3u^2\hat{u}+\hat{m}^2(\hat{u}^2+5u^2-6u\hat{u})}{64\hat{u}^3}\,, \\
\mathcal{F}_{2,1} &= -\frac{\hat{m}(u-\hat{u})(u^2-5\hat{u}m^2+u(3\hat{u}+m^2))}{64u^3\hat{u}}\,, \qquad (4.19) \\
\mathcal{F}_{2,2} &= \frac{u^4(5\hat{m}^2-3\hat{u})+u^3(11\hat{u}^2+25m^2\hat{m}^2-3\hat{u}(5m^2+11\hat{m}^2))+u^2(11\hat{u}^3+15\hat{u}m^2\hat{m}^2)}{1024u^3\hat{u}^3}+ \\
&\quad +\frac{-5u^2\hat{u}^2(m^2+\hat{m}^2)-3u(\hat{u}^4-5\hat{u}^2m^2\hat{m}^2+\hat{u}^3(11m^2+5\hat{m}^2))+5\hat{u}^3m^2(\hat{u}+5\hat{m}^2)}{1024u^3\hat{u}^3}\,, \qquad (4.20)
\end{aligned}
$$

where $u=-a_1a_2=a_1^2$ and $\hat{u}=-\hat{a}_1\hat{a}_2=\hat{a}_1^2$.

From these results it is possible to extract the results for the case with less matter hypermultiplets in the fundamental. This result is obtained just by taking m $(\hat{m})$ to infinity while keeping $m\Lambda_1$ $(\hat{m}\Lambda_2)$ fixed to the new scale. For example we have that for $\mathrm{SU}(2)\times\mathrm{SU}(2)$ without matter in the fundamental one gets

$$
\begin{aligned}
\mathcal{F}_{1,0} &= -\frac{(u+\hat{u})}{2u}\,, \qquad \mathcal{F}_{0,1} = -\frac{(u+\hat{u})}{2\hat{u}}\,, \qquad \mathcal{F}_{1,1} = \frac{(u+\hat{u})}{4u\hat{u}} \\
\mathcal{F}_{2,0} &= \frac{u^2+5\hat{u}^2-6u\hat{u}}{64u^3}\,, \qquad \mathcal{F}_{0,2} = \frac{\hat{u}^2+5u^2-6u\hat{u}}{64\hat{u}^3}\,, \\
\mathcal{F}_{2,1} &= -\frac{u^2+5\hat{u}^2-6u\hat{u}}{64u^3\hat{u}} \qquad (4.21)
\end{aligned}
$$

$\mathrm{SU}(3)\times\mathrm{SU}(2)$. In this example, for asymptotically free theories the most general case that one can consider is the case with two matter hypermultiplets in the fundamental of $\mathrm{SU}(3)$. We obtain the following results

$$
\begin{aligned}
\mathcal{F}_{1,0} &= \frac{-2u^3+2m_1m_2u^2-3(m_1+m_2)uv+18v^2-2u^2\hat{u}-6m_1m_2u\hat{u}+9(m_1+m_2)v\hat{u}}{4u^3-27v^2}\,, \\
\mathcal{F}_{0,1} &= -\frac{v}{2\hat{u}}\,, \\
\mathcal{F}_{2,0} &= \frac{5v^2+2u\hat{u}^2-3\hat{u}u^2+5v^2}{64\hat{u}^3}\,, \\
\mathcal{F}_{1,1} &= \frac{2\hat{u}(2u^2(m_1+m_2)-3uv)+2vu^2+6v(u-3\hat{u})m_1m_2-9v^2(m_1+m_2)}{2\hat{u}(4u^3-27v^2)}\,, \qquad (4.22)
\end{aligned}
$$

where $u=-a_1a_2-a_1a_3-a_2a_3$, $v=a_1a_2a_3$ and $\hat{u}=-\hat{a}_1\hat{a}_2=\hat{a}_1^2$.

As in the previous case, from these calculations it is possible to extract the results for the case with less matter in the fundamental. This result is obtained by taking m_1, m_2 to infinity while keeping $m_{1,2}\Lambda_1$ fixed to the new scale.

4.3 Some checks

In this subsection we will perform some non-trivial checks to the results obtained previously. The first check we can perform is to compare our results with the ones presented in [13] for the first instanton correction to the prepotential. Note that the first instanton correction obtained by us in (4.18) can be rewritten in the following way.[2]

$$\mathcal{F}_{1,0} = -\sum_{i=1}^{N_1} a_i \Delta_i^{(k)} = \sum_{i=1}^{N_1} \frac{P_0(a_i)P_2(a_i)}{\prod_{k\neq i}(a_i - a_k)^2}\,, \tag{4.24}$$

and an analogous expression for $\mathcal{F}_{0,1}$. Then, once we take $\hat{m}_k = 0$ ($m_k = 0$), we get the same result for the first instanton correction as the one in [13] for an $\mathrm{SU}(N_1)\times\mathrm{SU}(N_2)$ with massless hypermultiplets in the fundamental of both groups.

For higher instanton corrections the only result available in the literature is the one in [26] for the case $\mathrm{SU}(2)\times\mathrm{SU}(2)$. They compute up to the third instanton correction using the fact that for this particular case the curve is still hyperelliptic, so one can use the standard Picard-Fuchs techniques to calculate the instanton corrections. We have checked that our results for that case listed in (4.21) agree with those presented in [26].

Also, as we explained in the previous sections, another check that we can perform is that in the limit $\Lambda_2 \to 0$ one should recover the instanton corrections for a $\mathcal{N}=2$ supersymmetric Yang-Mills theory with gauge group $\mathrm{SU}(N_1)$ and $N_{f_1}+N_2$ matter hypermultiplets in the fundamental representation of the gauge group. In fact in that case we have

$$a_i = e_i + \sum_{k=1}^{\infty}\left(\frac{\partial}{\partial e_i}\right)^{2k-1}\Delta_i^{(k)}(e_i)\Lambda_1^{\beta_1}\,, \tag{4.25}$$

$$\Delta_i^{(k)}(x) = \frac{1}{(k!)^2}\left(\frac{\prod_{f=1}^{N_{f_1}}(x+m_f)\prod_{l=1}^{N_2}(x+\hat{e}_l)}{\prod_{j\neq k}(x-e_j)^2}\right)^k\,, \tag{4.26}$$

where now the quantities $\hat{a}_i$ play the role of masses in the fundamental of $\mathrm{SU}(N_1)$, so they do not have Λ_1 corrections and they are just given by $\hat{a}_i = \hat{e}_i$. From here we have

$$\begin{aligned}
\mathcal{F}_{1,0} &= -\sum_{i=1}^{N_1} a_i\Delta_i^{(1)}\,,\\
\mathcal{F}_{2,0} &= -\frac{1}{2}\sum_{i=1}^{N_1}\left(a_i\Delta_i^{(2)} + \frac{1}{2}\Delta_i^{(1)}\Delta_i^{(1)} + \partial_{a_i}\mathcal{F}_{1,0}\Delta_i^{(1)}\right),
\end{aligned}$$

[2] Using

$$\sum_{i=1}^{N_1}\mathrm{res}_{e_i}\, x\frac{P_0(x)P_2(x)}{(P_1(x))^2} = \sum_{i=1}^{N_1} e_i\Delta_i^{(k)}(e_i) + \sum_{i=1}^{N_1}\frac{P_0(e_i)P_2(e_i)}{\prod_{k\neq i}(e_i-e_k)^2} = 0\,, \tag{4.23}$$

up to constant unphysical terms that come from the residue at infinity of the function $x\frac{P_0(x)P_2(x)}{(P_1(x))^2}$.

$$
\begin{aligned}
\mathcal{F}_{3,0} &= -\frac{1}{3}\sum_{i=1}^{N_1}\Bigg(a_i\Delta_i^{(3)} + \Delta_i^{(1)}\Delta_i^{(2)} + 2\partial_{a_i}\mathcal{F}_{2,0}\Delta_i^{(1)} + \partial_{a_i}\mathcal{F}_{1,0}\Delta_i^{(2)} + \\
&\qquad + \sum_{j=1}^{N_1}\frac{1}{2}\partial_{a_i}\partial_{a_j}\mathcal{F}_{1,0}\Delta_i^{(1)}\Delta_j^{(1)}\Bigg)\,, \\
&\vdots
\end{aligned}
\tag{4.27}
$$

that are exactly the instanton corrections for a $\mathcal{N}=2$ supersymmetric Yang-Mills theory with gauge group $\mathrm{SU}(N_1)$ and $N_{f_1}+N_2$ matter hypermultiplets in the fundamental representation of the gauge group, as one would expect. Also the same is true when $\Lambda_2 \to 0$. This limit fixes completely the first instanton correction to the prepotential. In fact, the result (4.27) agrees with the results available in the literature for these cases (see for example [16, 20, 21, 22]).

Another non-trivial check of the result that we can perform is that in the case $N_1 = N_2 = N$, in the limit $\Lambda_1, \Lambda_2 \to 0$, $\Lambda_1\Lambda_2 \to \Lambda^2$ and $a_i = \hat{a}_i$ one should recover the result for $\mathrm{SU}(N)$ with $N_{f_1}+N_{f_2}$ matter hypermultiplets in the fundamental representation. Taking this limit in (4.20) we get

$$
\begin{aligned}
\mathcal{F}_{1,1} &\longrightarrow \frac{m\hat{m}}{2u} \equiv \mathcal{F}_1^{\mathrm{SU}(2)}\,, \\
\mathcal{F}_{2,2} &\longrightarrow \frac{u^2 - 3u(m^2+\hat{m}^2) + 5m^2\hat{m}^2}{64u^3} \equiv \mathcal{F}_2^{\mathrm{SU}(2)}\,,
\end{aligned}
$$

where $u = -a_1 a_2 = a_1^2$. Those are exactly the expected results.

5. Generalization to higher representations of the gauge group

In this section we will use the technique developed above to calculate the instanton corrections to the prepotential for $\mathcal{N}=2$ super Yang-Mills theory with a matter hypermultiplet in the symmetric and in the antisymmetric representation of the gauge group $\mathrm{SU}(N)$.

5.1 The symmetric representation

The curve for this case is [10]

$$
y^3 + P(x)y^2 + x^2P(-x)\Lambda^{N-2} + x^6\Lambda^{3(N-2)} = 0\,, \tag{5.1}
$$

where $P(x) = \prod_{i=1}^{N}(x-e_i)$ denotes the characteristic polynomial of $\mathrm{SU}(N)$. Also we have $\sum e_i = \frac{N}{2}m$, m being the mass of the hypermultiplet in the symmetric representation. Therefore, this curve has the same form as the curve (2.1) just by identifying $P_0(x)=1$, $P_1(x) = P(x)$, $P_2(x) = x^2P(-x)$ and $P_3(x) = x^6$. Also one should take $\Lambda_1 = \Lambda_2 = \Lambda$.

It is easy to see that, for this case eq. (3.15) takes the form

$$
\frac{1}{2}\sum_{i=1}^{N} a_i^2 + \sum_{k=1}^{\infty} k\mathcal{F}_k(a_i)\Lambda^{k(N-2)} = \frac{1}{2}\sum_{i=1}^{N} e_i^2\,, \tag{5.2}
$$

where

$$a_i = e_i + \sum_{k=1}^{\infty} \Delta_i^{(k)}(e_i)\Lambda^{k\beta} + \sum_{k,l=1}^{\infty} \Delta_i^{(k,l)}(e_i)\Lambda^{\beta(k+3l-1)}\,, \tag{5.3}$$

$$\Delta_i^{(k)}(x) = \frac{1}{(k!)^2}\left(\frac{\partial}{\partial x}\right)^{2k-1}\left(\frac{x^{2k}P(-x)^k}{\prod_{j\neq i}(x-e_j)^{2k}}\right), \tag{5.4}$$

$$\Delta_i^{(k,1)}(x) = -\frac{k}{k!(k+1)!}\left(\frac{\partial}{\partial x}\right)^{2k}\left(\frac{x^6(P(-x))^{k-1}}{\prod_{j\neq i}(x-e_j)^{2k+1}}\right), \tag{5.5}$$

$$\vdots$$

Therefore the instanton corrections have the following the form

$$\begin{aligned}
\mathcal{F}_1(a_i) &= -\sum_{i=1}^{N} a_i\Delta_i^{(1)}\,,\\
\mathcal{F}_2(a_i) &= -\frac{1}{2}\sum_{i=1}^{N}\left(a_i\Delta_i^{(2)} + \frac{1}{2}\Delta_i^{(1)}\Delta_i^{(1)} + \partial_{a_i}\mathcal{F}_1\Delta_i^{(1)}\right),\\
\mathcal{F}_3(a_i) &= -\frac{1}{3}\sum_{i=1}^{N}\Bigg(a_i(\Delta_i^{(3)} + \Delta_i^{(1,1)}) + \Delta_i^{(1)}\Delta_i^{(2)} + 2\partial_{a_i}\mathcal{F}_2\Delta_i^{(1)} + \partial_{a_i}\mathcal{F}_1\Delta_i^{(2)} + \\
&\qquad + \sum_{j=1}^{N}\frac{1}{2}\partial_{a_i}\partial_{a_j}\mathcal{F}_1\Delta_i^{(1)}\Delta_j^{(1)}\Bigg),\\
&\vdots
\end{aligned} \tag{5.6}$$

where to recover the mass dependence one should shift $a_i \to a_i + \frac{m}{2}$.

Note that by the same calculation as in (4.23) we obtain

$$\mathcal{F}_1 = -\sum_{i=1}^{N} a_i\Delta_i^{(1)} = \sum_{i=1}^{N}\frac{a_i^2P(-a_i)}{\prod_{k\neq i}(a_i-a_k)^2}\,, \tag{5.7}$$

so we get the same result as in [12] for the first instanton correction. The first instanton correction for this case has been also computed in [27] using the ADHM instanton calculus, and we find also a pefect agreement with those results. There are no results available in the literature for higher instanton corrections.

5.2 The antisymmetric representation

The curve for this case is [10]

$$y^3 + \left(x^2P(x) + 3\Lambda^{N+2}\right)y^2 + \left(x^2P(-x) + 3\Lambda^{N+2}\right)\Lambda^{N+2} + \Lambda^{3(N+2)} = 0\,, \tag{5.8}$$

where $P(x) = \prod_{i=1}^{N}(x-e_i)$ denotes the characteristic polynomial of SU(N). Also we have $\sum e_i = -\frac{N}{2}m$, m being the mass of the hypermultiplet in the antisymmetric representation. Therefore, this curve has the same form as the curve (2.1) just by identifying $P_0(x) = 1$,

$P_1(x) = x^2 P(x) + 3\Lambda^{N+2}$, $P_2(x) = x^2 P(-x) + 3\Lambda^{N+2}$ and $P_3(x) = 1$. Also one should take $\Lambda_1 = \Lambda_2 = \Lambda$. This case is a bit more complicated than the former ones because there is dependence in Λ in P_1 and P_2 and therefore one has to be more careful in the computation of the series expansion of dS_{SW}. In any case, one can still obtain the instanton corrections to the prepotential using the same procedure.

In fact, for the first terms of the series expansion of a_i around $\Lambda = 0$ we have

$$a_i = e_i + \sum_{\substack{m,n\geq 0 \\ m+n>0}}^{N} \Delta_i^{(m,n)}(e_i)\Lambda^{\beta(m+n)} + \dots\,, \tag{5.9}$$

where

$$\Delta_i^{(m,n)}(x) = \frac{(-3)^n}{(m!)^2 n!}\left(\frac{\partial}{\partial x}\right)^{2m+n-1} \frac{(x^2 P(-x) + 3\Lambda^\beta)^m}{x^{4m+2n}\prod_{k\neq i}(x-e_k)^{2m+n}}\Lambda^{\beta(m+n)}\,, \tag{5.10}$$

being expression (5.9) exact up to order $\Lambda^{3(N+2)}$.

Also here eq. (3.15) takes the form

$$\frac{1}{2}\sum_{i=1}^{N} a_i^2 + \sum_{k=1}^{\infty} k\mathcal{F}_k(a_i)\Lambda^{k(N+2)} = \frac{1}{2}\sum_{i=1}^{N} e_i^2\,, \tag{5.11}$$

from where we can extract the instanton corrections to the prepotential. These are given by

$$\mathcal{F}_1 = -\sum_{i=1}^{N} a_i \partial_x \frac{P(-x)}{x^2\prod_{k\neq i}(x-a_k)^2}\Bigg|_{x=a_i} + \sum_{i=1}^{N}\frac{3}{a_i\prod_{k\neq i}(a_i-a_k)}\,, \tag{5.12}$$

$$\begin{aligned}\mathcal{F}_2 = &-\frac{1}{2}\sum_{i=1}^{N}\Bigg(a_i\Bigg(\frac{15}{2}\partial_x\frac{1}{x^4\prod_{k\neq i}(x-a_k)^2}\Bigg|_{x=a_i} - 3\partial_x\frac{P(-x)}{x^4\prod_{k\neq i}(x-a_k)^3}\Bigg|_{x=a_i} + \\ &\qquad + \frac{1}{4}\partial_x\frac{(P(-x))^2}{x^4\prod_{k\neq i}(x-a_k)^4}\Bigg|_{x=a_i}\Bigg) + \\ &+\frac{1}{2}\left(\partial_x\frac{P(-x)}{x^2\prod_{k\neq i}(x-a_k)^2}\Bigg|_{x=a_i} - \frac{3}{a_i^2\prod_{k\neq i}(a_i-a_k)}\right)^2 + \\ &+\partial_{a_i}\mathcal{F}_1\left(\partial_x\frac{P(-x)}{x^2\prod_{k\neq i}(x-a_k)^2}\Bigg|_{x=a_i} - \frac{3}{a_i^2\prod_{k\neq i}(a_i-a_k)}\right)\Bigg)\,, \\ &\vdots\end{aligned} \tag{5.13}$$

where to recover the mass dependence one should shift $a_i \to a_i - \frac{m}{2}$.

Note that now we have

$$\begin{aligned}\sum_{i=1}^{N}\mathrm{res}_{e_i} x\frac{P(-x)}{(xP(x))^2} &= \sum_{i=1}^{N}\frac{P(-e_i)}{e_i^2\prod_{k\neq i}(e_i-e_k)^2} + \sum_{i=1}^{N} e_i\partial_x\frac{P(-x)}{x^2\prod_{k\neq i}(x-e_k)^2}\Bigg|_{x=e_i} = -\frac{1}{P(0)}\,, \\ \sum_{i=1}^{N}\mathrm{res}_{e_i}\frac{3}{xP(x)} &= \sum_{i=1}^{N}\frac{3}{e_i\prod_{k\neq i}(e_i-e_k)} = -\frac{3}{P(0)}\,,\end{aligned} \tag{5.14}$$

where the last term in both equations in (5.14) comes from the fact that the functions $x\frac{P(-x)}{(xP(x))^2}$ and $3/xP(x)$, have also a non-vanishing residue in $x = 0$, not just in $x = e_i$. Therefore, we have

$$\mathcal{F}_1 = \sum_{i=1}^{N} \frac{P(-a_i)}{a_i^2 \prod_{k\neq i}(a_i - a_k)^2} - \frac{2}{P(0)} \,. \tag{5.15}$$

Note that this result agrees with the one presented in previous calculations [11]. There are no results available in the literature for higher instanton corrections.

We can also check this result if we take into account the fact that SU(2) with matter in the antisymmetric should give us the same result as pure SU(2). In fact what we find is

$$\mathcal{F}_1 = \frac{2}{u} \,, \qquad \mathcal{F}_2 = \frac{5}{4u^3} \,, \tag{5.16}$$

where $u = -a_1 a_2 = a_1^2$. This is exactly the result for pure SU(2) obtained for example in [20, 21] with a change in the quantum scale $\Lambda^4 \to \frac{\Lambda^4}{4}$. Also for SU(3) with one hypermultiplet in the antisymmetric we should get the same result as SU(3) with one hypermultiplet in the fundamental. In fact we get

$$\mathcal{F}_1 = 4\frac{6um - 9v}{4u^3 - 27v^2} \,, \tag{5.17}$$

where $u = -a_1 a_2 - a_1 a_3 - a_2 a_3$ and $v = a_1 a_2 a_3$. This is exactly the result for SU(3) with one hypermultiplet in the fundamental representation obtained for example in [20, 21], also with a change in the quantum scale $\Lambda^4 \to \Lambda^4/4$.

6. Conclusions

In the present paper we have studied the form of the prepotential of $\mathcal{N} = 2$ supersymmetric gauge theories with gauge group $\mathrm{SU}(N_1) \times \mathrm{SU}(N_2)$ with a hypermultiplet in the bifundamental representation and matter in the fundamental representation of both gauge groups. The Seiberg-Witten curves for those theories are non-hyperelliptic curves derived from M-theory considerations. In the first place we calculate the logarithmic derivatives of the prepotential with respect to the quantum scales of both gauge groups. With the help of the Riemann bilinear relations we express this logarithmic derivative in terms of the moduli of the Seiberg-Witten curve. In fact we find that this logarithmic derivative of the prepotential with respect to the quantum scales Λ_1 and Λ_2 shows a non-trivial mixing between both groups order parameters, as is expected in the presence of a hypermultiplet in the bifundamental representation. As an application we develop a method to compute recursively the instanton corrections to the prepotential with the help of the previously calculated equation. Using this, we find that we can compute the instanton corrections recursively in a straightforward way. This improves the existing method developed in [11]–[14]. In that references they compute the first instanton correction calculating explicitly the dual order parameters and integrating them. That method gets very complicated for higher corrections, so does not allow one to go further. The method described here is much simpler as we avoid the calculation of the dual periods. We also extend the method to compute the instanton corrections to the non-hyperelliptic curves obtained for SU(N) theories with matter in the symmetric and antisymmetric representation, finding also that it allows us to compute recursively the instanton corrections to the prepotential.

It is important to note that for the case of pure SU(N), or with matter in the fundamental representation, the Seiberg-Witten curves were originally calculated using just field theory considerations. Therefore the comparison of the results obtained using the Seiberg-Witten curves with the ones obtained from a microscopic calculation (or from other alternative methods, like the one presented in [28]) are seen as tests of the Seiberg-Witten approach. Nevertheless, the Seiberg-Witten curves for supersymmetric gauge theories with product gauge groups, or matter in the symmetric or antisymmetric representation of the gauge group, are non-hyperelliptic curves derived just from M-theory considerations. Therefore, when microscopic calculations will become available for these theories, the comparison of results is not just a test of the Seiberg-Witten approach but also from the M-theory considerations used to derive the Seiberg-Witten curves.

Also we must point out that the recent papers of Dijkgraaf and Vafa [29] have attracted a new attention to the subject of Seiberg-Witten theory. Their work indicates that several non-perturbative results in supersymmetric gauge theories can be obtained by means of perturbative calculations using auxiliary matrix models. It would be interesting to see if the results obtained here can be reproduced from the matrix model approach.

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